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ASYMPTOTIC NORMALITY OF A VARIANCE ESTIMATOR OF A LINEAR COMBINATION OF A FUNCTION OF ORDER STATISTICS

By

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Abstract

An estimator of the asymptotic variance of (a randomly stopped) linear combination of a function of order statistics is considered and its asymptotic normality is studied under appropriate regularity conditions. A comparative study of the regularity conditions pertaining to the asymptotic normality and strong convergence of linear combinations of functions of order statistics and their estimated asymptotic variances is also made.

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1. Introduction.

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d.r.v.) with a continuous distribution function (df) F, defined on the real line $R = (-\infty,\infty)$. For every n (≥ 1), let $X_{n,1},\ldots,X_{n,n}$ be the order statistics corresponding to $X_1,\ldots,X_n$ and consider the statistics

$$T_{n,k} = n^{-1} \Sigma_{i=1}^{k} c_{n,i} h(X_{n,i}), \quad 1 \leq k \leq n, \tag{1.1}$$

where $\{c_{n,i}, 1 \leq i \leq n; n \geq 1\}$ is a triangular array of (known) real constants and h is a specified function. Actually, if we let $g = h \circ F^{-1}$ and $\xi_{n,i} = F(X_{n,i})$, $1 \leq i \leq n$ (so that $\xi_{n,1},\ldots,\xi_{n,n}$ are the ordered r.v. of a sample of size n from the uniform $(0,1)$ df), we may rewrite (1.1) as

$$T_{n,k} = n^{-1} \Sigma_{i=1}^{k} c_{n,i} g(\xi_{n,i}), \quad 1 \leq k \leq n. \tag{1.2}$$

Under suitable regularity conditions (on g and the $c_{n,i}$), for $k/n \to \alpha$ $(0 < \alpha \leq 1)$,

$$n^{\frac{1}{2}}(T_{n,k} - \mu(\alpha))/\sigma(\alpha) \underset{\mathcal{L}}{\to} N(0,1), \tag{1.3}$$

where for each $\alpha \in (0,1]$, $\mu(\alpha)$ (asymptotic mean) and $\sigma^2(\alpha)$ (asymptotic variance) are functionals of g and the score function J (which generates the $c_{n,i}$). (1.3) has been proved under diverse regularity conditions by a host of research workers (viz. [1, 3, 4, 5, 6, 7, 8]). Stigler (1969) has also shown that under suitable regularity conditions,

$$n \operatorname{Var}(T_{n,k})/\sigma^2(\alpha) \to 1 \quad \text{as} \quad n \to \infty . \tag{1.4}$$

Let $\{\tau_n, n \geq 1\}$ be a class of stopping times, where, for each n (≥ 1), τ_n is defined in terms of $X_{n,1},\ldots,X_{n,n}$ and it assumes values in

$\{1,\ldots,n\}$. Gardiner and Sen (1978) have shown that if $n^{-1}\tau_n \xrightarrow{p} \alpha \in (0,1]$ and the regularity conditions pertaining to (1.3) hold, then

$$n^{\frac{1}{2}}(T_{n,\tau_n} - \mu(n^{-1}\tau_n))/\sigma(\alpha) \underset{\mathcal{L}}{\rightarrow} N(0,1), \tag{1.5}$$

while if $n^{-\frac{1}{2}}(\tau_n - n\alpha) \xrightarrow{p} 0$, then in (1.5), $\mu(n^{-1}\tau_n)$ may also be replaced by $\mu(\alpha)$.

In a variety of practical applications, $\mu(\alpha)$ can be related to the basic (viz., location or scale) parameters of F, and thereby, confidence intervals or tests of significance for $\mu(\alpha)$ can be transmitted to yield parallel conclusions for these parameters. In this context, one confronts the problem of estimating $\sigma^2(\alpha)$ and natural estimators of $\sigma^2(\alpha)$ can be derived from the sample. The object of the present investigation is to consider such an estimator of $\sigma^2(\alpha)$ and to study its asymptotic normality. Along with the preliminary notions, the main theorems are presented in Section 2 and their proofs are considered in Section 3. Section 4 is devoted to some general remarks including a comparative study of the regularity conditions pertaining to the almost sure (a.s.) convergence and asymptotic normality of $T_{n,k}$ and the estimator of $\sigma^2(\alpha)$. For the convenience of presentation, some of the technicalities are postponed to the Appendix.

2. Preliminary notions and the main theorems.

Define g as in after (1.1) and assume that for every $\theta \in (0,\frac{1}{2})$, g is of bounded variation in $(\theta, 1-\theta)$. For each n (≥ 1), define J_n on $[0,1]$ by letting $J_n(t) = c_{n,i}$ for $(i-1)/n < t \leq i/n$, $1 \leq i \leq n$ and $J_n(0) = c_{n,1}$. Also, let $\Gamma_n(t) = n^{-1}\Sigma_{i=1}^{n} I(\xi_{n,i} \leq t)$, $t \in [0,1]$ be the empirical df. Then $T_{n,k}$ in (1.2) can be expressed as

$$T_{n,k} = \int_0^{\Gamma_n^{-1}(\frac{k}{n})} J_n(\Gamma_n(t))g(t)d\Gamma_n(t). \tag{2.1}$$

We define a bounding function

$$B(\cdot, \underset{\sim}{a}) = \{B(t, \underset{\sim}{a}) = M\, t^{-a_1}(1-t)^{-a_2},\ t \in (0,1)\} \tag{2.2}$$

where $M(0 < M < \infty)$, $\underset{\sim}{a} = (a_1, a_2)$ and a_1, a_2 are real numbers. Also, for fixed $\beta(> 0)$ and $\delta(> 0)$, we define

$$q_\beta = \{q_\beta(t) = [t(1-t)]^{\beta-\delta/2},\ t \in (0,1)\}. \tag{2.3}$$

Then, we make the following assumptions:

[A1]: $|g| \leq B(\cdot, \underset{\sim}{a})$ for some $\underset{\sim}{a} = (a_1, a_2)$.

[A2]: There exists a J, defined on $(0,1)$, such that

$$|J| \leq B(\cdot, \underset{\sim}{b}) \text{ and } |J_n| \leq B(\cdot, \underset{\sim}{b}),\ \forall n, \tag{2.4}$$

where $\underset{\sim}{b} = (b_1, b_2)$ with real b_1, b_2 and except on a set of t's of $|g|$-measure zero, both J is continuous at t and $J_n \to J$ uniformly in some neighborhood of t as $n \to \infty$.

For each $\alpha \in (0,1]$, let us then define

$$\mu_n(\alpha) = \int_0^\alpha J_n(t)g(t)dt, \tag{2.5}$$

$$\sigma^2(\alpha) = \int_0^\alpha \int_0^\alpha (s \wedge t - st)J(s)J(t)dg(s)dg(t); \quad a \wedge b = \min(a,b). \tag{2.6}$$

Note that if

$$a_1 + b_1 = a_2 + b_2 = 1/2 - \delta \tag{2.7}$$

then $\int_0^1 B(\cdot, \underset{\sim}{b})q_{\frac{1}{2}}d|g| < \infty$ and it follows from assumptions A1, A2 that both $\mu_n(\alpha)$ and $\sigma^2(\alpha)$ are finite and then (1.3) holds [cf. Shorack (1972)]. If, in addition $n^{-1}\tau_n \overset{p}{\to} \alpha \in (0,1)$ and g admits a derivative

at α or $n^{-\frac{1}{2}}(\tau_n - n\alpha) = 0_p(1)$ and g is continuous at α then (1.5) obtains [cf. Gardiner & Sen (1978)].

In the current paper, we consider the following estimator of $\sigma^2(\alpha)$:

$$\hat{\sigma}_n^2(\alpha) = \int_0^\alpha \int_0^\alpha \{\Gamma_n(s \wedge t) - \Gamma_n(s)\Gamma_n(t)\}J_n(\Gamma_n(s))J_n(\Gamma_n(t))dg(s)dg(t) \quad (2.8)$$

which can also be written as

$$\hat{\sigma}_n^2(\alpha) = n^{-2}\Sigma_{i=1}^{n^*-1} \Sigma_{j=1}^{n^*-1} c_{n,i}c_{n,j}[n(i \wedge j) - ij][h(X_{n,i+1}) - h(X_{n,i})][h(X_{n,j+1}) - h(X_{n,j})] + r_n, \quad (2.9)$$

where $n^* = \max\{k: \xi_{n,k} \leq \alpha\}$ and $r_n = 0_p(n^{-1})$. Also, as in Sen (1978), $\hat{\sigma}_n^2(\alpha)$ can be interpreted as the conditional variance of nT_{n,n^*} given $\{X_{n+k,j}, 1 \leq j \leq n+k$ and $k \geq 1\}$. Our main concern is to study regularity conditions pertaining to the asymptotic normality of $n^{\frac{1}{2}}(\hat{\sigma}_n^2(\alpha) - \sigma^2(\alpha))$. For this purpose we need some additional regularity conditions:

[A3]: $n^{\frac{1}{2}} \int_0^1 |J_n(\Gamma_n(t)) - J(\Gamma_n(t))|d|g(t)| \overset{p}{\to} 0$, as $n \to \infty$,

[A4]: except on a set of t's of $|g|$ measure zero, $J'(t) = (d/dt)J(t)$ exists and is continuous at t, and for some $\underset{\sim}{c} = (c_1, c_2)$,

$$|J'| \leq B(\cdot, \underset{\sim}{c}) \text{ where } 0 \leq c_1 - b_1, c_2 - b_2 \leq 1, \quad (2.10)$$

with $\underset{\sim}{b}$ defined in [A2].

Let us now write I for the identity function on $(0,1)$ and let

$$J_{(1)} = IJ, \quad J_{(2)} = (1 - I)J; \quad (2.11)$$

$$L_1(t) = 2\int_t^1 J_{(2)}dg, \quad L_2(t) = 2\int_0^t J_{(1)}dg, \quad 0 < t < 1; \quad (2.12)$$

$$L_0 = L_1J'_{(1)} + L_2J'_{(2)}. \quad (2.13)$$

Define

$$\gamma^2 = \int_0^1 \int_0^1 (s \wedge t - st) L_0(s) L_0(t) dg(s) dg(t). \tag{2.14}$$

Then, we have the following.

Theorem 1. Suppose that A1, A2, A3 and A4 hold and

$$\int_0^1 B(\cdot, \underset{\sim}{b}) q_4 d|g| < \infty. \tag{2.15}$$

Then, both $\sigma^2(1)$ and γ^2 are finite and

$$n^{\frac{1}{2}}(\hat{\sigma}_n^2(1) - \sigma^2(1))/\gamma \underset{L}{\to} N(0,1). \tag{2.16}$$

The proof is considered in the next section. We may remark here that in (2.11) through (2.14), if we let $J(t) = 0$ for $t \geq \alpha$ (when $0 < \alpha < 1$) and denote the resulting expression in (2.14) by γ_α^2, then (2.16) holds for $n^{\frac{1}{2}}(\hat{\sigma}_n^2(\alpha) - \sigma^2(\alpha))/\gamma_\alpha$. Hence, for the sake of simplicity, we consider the case of $\alpha = 1$ and, for notational convenience, write $\hat{\sigma}_n^2(1) = \hat{\sigma}_n^2$, $\sigma^2(1) = \sigma^2$. We may also remark that whenever L_0 in (2.13) is integrable with respect to the signed measure g on $(0,1)$, a more convenient form of (2.14) can be obtained. Define G_0 on $(0,1)$ by

$$G_0(t) = \int_0^t L_0(s) dg(s), \ 0 < t < 1. \tag{2.17}$$

Then, a pedestrian calculation leads us to

$$\gamma^2 = \int_0^1 G_0^2(t) dt - \left(\int_0^1 G_0(t) dt\right)^2 . \tag{2.18}$$

Now let us suppose $0 < \alpha < 1$ and set $\sigma_n^{*2} = \hat{\sigma}_n^2(n^{-1} \tau_n)$. In the statement of A4 we assume additionally that J is continuous at α and $J_n \to J$ uniformly in some neighborhood of α as $n \to \infty$. For $t \in (0,\alpha)$ we define $L_1^+(t) = 2\int_t^\alpha J_{(2)} dg$ and $L_0^+ = L_1^+ J'_{(1)} + L_2 J'_{(2)}$. Let $(\gamma^+(\alpha))^2 = \int_0^\alpha \int_0^\alpha (s \wedge t - st) L_0^+(s) L_0^+(t) dg(s) dg(t)$.

Theorem 2. With the remarks noted above suppose that A1 through A4 hold together with (2.15). Then both $\sigma^2(\alpha)$ and $\gamma^+(\alpha)$ are finite and if, in addition, $n^{-1}\tau_n \xrightarrow{p} \alpha$ then

$$n^{\frac{1}{2}}(\sigma_n^{*2} - \sigma^2(n^{-1}\tau_n))/\gamma^+(\alpha) \xrightarrow{\mathcal{L}} N(0,1) \tag{2.19}$$

while if $n^{-\frac{1}{2}}(\tau_n - n\alpha) \xrightarrow{p} 0$ and g admits a derivative at α then in (2.19) $\sigma^2(n^{-1}\tau_n)$ may also be replaced by $\sigma^2(\alpha)$.

3. Proofs of Theorems.

Note that by (2.4) and (2.6),

$$\begin{aligned}
0 \le \sigma^2(\alpha) &= 2 \iint_{0<s<t<\alpha} s(1-t)J(s)J(t)dg(s)dg(t) \\
&\le 2(\int_0^\alpha \{t(1-t)\}^{\frac{1}{2}}|J(t)|d|g(t)|)^2 \\
&\le 2(\int_0^1 \{t(1-t)\}^{\frac{1}{2}}|J(t)|d|g|)^2, \ \forall \alpha \in (0,1] \\
&\le 2M^2(\int_0^1 Bq_{\frac{1}{2}}d|g|)^2, \ \forall \alpha \in (0,1].
\end{aligned} \tag{3.1}$$

Now (2.15) ensures the less restrictive condition $\int_0^1 Bq_{\frac{1}{2}}d|g| < \infty$ and so $\sigma^2(\alpha) < \infty$ for every $\alpha \in (0,1]$. Similarly, on noting that under (2.15), by (2.11), (2.12) and (2.13)

$$|L_0(t)| \le M^*\{t(1-t)\}^{-\frac{1}{2}}B(t, \underset{\sim}{b}), \ \forall t \in (0,1), \text{ for some } M^* < \infty, \tag{3.2}$$

we have by (2.14) and (3.2),

$$\begin{aligned}
\gamma^2 &\le 2(\int_0^1 \{t(1-t)\}^{\frac{1}{2}}|L_0(t)|d|g(t)|)^2 \\
&\le 2(M^* \int_0^1 \{t(1-t)\}^{\frac{1}{2}}B(t, \underset{\sim}{b})d|g(t)|)^2 \\
&< 2M^{*2}(\int B(\cdot, \underset{\sim}{b})q_{\frac{1}{4}}d|g|)^2 < \infty, \text{ by } (2.15)
\end{aligned} \tag{3.3}$$

Note that by (2.6), (2.11) and (2.12),

$$\sigma^2 = 2 \iint_{0<s<t<1} s(1-t)J(s)J(t)dg(s)dg(t) = \frac{1}{2}\int_0^1 L_1(t)dL_2(t). \qquad (3.4)$$

Again, if we define for each $n > 1$ and $t \in [\xi_{n,1}, \xi_{n,n})$,

$$L_{n,1}(t) = 2\int_t^{\xi_{n,n}} (1-\Gamma_n)J_n(\Gamma_n)dg \quad \text{and} \quad L_{n,2}(t) = 2\int_{\xi_{n,1}}^t \Gamma_n J_n(\Gamma_n)dg, \qquad (3.5)$$

with both $L_{n,1}$ and $L_{n,2}$ set equal to zero otherwise, we may write

$$\hat{\sigma}_n^2 = (1/2)\int_0^1 L_{n,1}(t)dL_{n,2}(t). \qquad (3.6)$$

From (3.4) and (3.6), we have

$$n^{\frac{1}{2}}(\hat{\sigma}_n^2 - \sigma^2) = \frac{1}{2}\{S_{n,1} + S_{n,2} + R_n\}, \qquad (3.7)$$

where

$$S_{n,1} = \int_0^1 n^{\frac{1}{2}}(L_{n,1}(t) - L_1(t))dL_2(t), \qquad (3.8)$$

$$S_{n,2} = \int_0^1 L_1(t)d\{n^{\frac{1}{2}}(L_{n,2}(t) - L_2(t))\}, \qquad (3.9)$$

$$R_n = \int_0^1 n^{\frac{1}{2}}(L_{n,1}(t) - L_1(t))d(L_{n,2}(t) - L_2(t)). \qquad (3.10)$$

Let $(\Omega,\mathcal{B},P)$ be the underlying probability space and let $U_n = n^{\frac{1}{2}}(\Gamma_n - I)$ be the uniform empirical process on $[0,1]$. Suppose U denotes a standard Brownian bridge process on $[0,1]$ defined on the same probability space. [$(\Omega,\mathcal{B},P)$ may not be rich enough to support U. However, by one of the usual techniques of embedding [cf. Shorack (1972)], we may construct another probability space $(\Omega^*,\mathcal{B}^*,P^*)$ where the distributions of our original variables are preserved and which is rich enough to support U.] Let Q be the class of all nonnegative, continuous q on $[0,1]$ which are bounded below by functions $\bar{q}$ nondecreasing (nonincreasing) on $[0,\frac{1}{2}]$ ($[\frac{1}{2},1]$) and

satisfy $\int_0^1 \bar{q}^{-2} dI < \infty$. Let $\rho_q(f,g) = \sup\{|f(t)-g(t)|/q(t): 0 < t < 1\}$ be the usual sup-norm metric. Then, it is known that for each $q \in Q$

$$\rho_q(U_n, U) = o_p(1) \quad \text{and} \quad \rho_q(U_n, 0) = O_p(1) = \rho_q(U, 0). \tag{3.11}$$

Note that by our definitions,

$$\begin{aligned} \tfrac{1}{2} n^{\frac{1}{2}}(L_{n,1} - L_1) &= -\int_t^{\xi_{n,n}} U_n J_n(\Gamma_n) dg + \int_t^{\xi_{n,n}} (1-I) n^{\frac{1}{2}}(J_n(\Gamma_n) - J(\Gamma_n)) dg \\ &\quad + \int_t^{\xi_{n,n}} (1-I) n^{\frac{1}{2}}(J(\Gamma_n) - J) dg - n^{\frac{1}{2}} \int_{\xi_{n,n}}^1 (1-I) J dg, \\ &\qquad\qquad \text{for } t \in [\xi_{n,1}, \xi_{n,n}) \\ &= -n^{\frac{1}{2}} \int_t^1 (1 - I) J dg, \text{ otherwise} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \tfrac{1}{2} n^{\frac{1}{2}}(L_{n,2} - L_2) &= \int_{\xi_{n,1}}^t U_n J_n(\Gamma_n) dg + \int_{\xi_{n,1}}^t I n^{\frac{1}{2}}(J_n(\Gamma_n) - J(\Gamma_n)) dg \\ &\quad + \int_{\xi_{n,1}}^t I\, n^{\frac{1}{2}}(J(\Gamma_n) - J) dg - n^{\frac{1}{2}} \int_0^{\xi_{n,1}} I\, J dg, \\ &\qquad\qquad \text{for } t \in [\xi_{n,1}, \xi_{n,n}), \\ &= -n^{\frac{1}{2}} \int_0^t I J dg, \text{ otherwise.} \end{aligned} \tag{3.13}$$

Substituting (3.12) in (3.8), we write

$$S_{n,1} = -S_{n,1}^{(1)} + S_{n,1}^{(2)} + S_{n,1}^{(3)} - S_{n,1}^{(4)} . \tag{3.14}$$

Define $\zeta_1 = \int_0^1 \{\int_t^1 UJdg\} dL_2$ and let $\chi_{n,1}, \chi_{n,2}$ denote the indicators of $[\xi_{n,1}, \xi_{n,n})$ and $[t, \xi_{n,n})$, $t \in (0,1)$ respectively. Then

$$|S_{n,1}^{(1)} - \zeta_1| \le \int_0^1 IB(\cdot,\underset{\sim}{b})\chi_{n,1}d|g|\{\int_t^1 |\chi_{n,2}U_nJ_n(\Gamma_n) - UJ|d|g|\}$$

$$+ \int_0^1 IB(\cdot,\underset{\sim}{b})\bar{\chi}_{n,1}d|g|\{\int_t^1 |UJ|d|g|\} \tag{3.15}$$

$$= S_{n,11}^{(1)} + S_{n,12}^{(1)}, \text{ say,}$$

where $\bar{\chi}_{n,1}$ is the indicator of the complement of $[\xi_{n,1}, \xi_{n,n})$ relative to $(0,1)$. To handle $S_{n,11}^{(1)}$ note that for $\chi_{n,1} = 1$ and $\chi_{n,2} = 1$ we have

$$|\chi_{n,2}U_nJ_n(\Gamma_n) - UJ| \le |U_n - U|B(\cdot,\underset{\sim}{b}) + |J_n(\Gamma_n) - J||U_n| . \tag{3.16}$$

Furthermore, $|J_n(\Gamma_n) - J| \le 2B(\Gamma_n,\underset{\sim}{b}) \vee B(I,\underset{\sim}{b})$, and since $0 < \Gamma_n < 1$, in the range under consideration, we obtain by Theorem 2 of Wellner (1977) that there exists a set $A \subset \Omega$ such that $P(A) = 1$ and for each $\omega \in A$ there exists an integer n_ω for which $n \ge n_\omega$ implies

$$|J_n(\Gamma_n) - J| \le M^0 B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}}/\tilde{q} , \tag{3.17}$$

where M^0 $(< \infty)$ is a constant and $\tilde{q} = \{I(1 - I)\}^{\frac{1}{2}-\delta/4}$. For such ω and n, therefore, from (3.11) and (3.17), the right hand side of (3.16) is bounded by

$$\rho_{q_{\frac{1}{2}}}(U_n,U)B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}} + M^0\rho_{\tilde{q}}(U_n,0)B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}} = O_p(1)B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}} \tag{3.18}$$

whenever $\chi_{n,1} = 1$ and $\chi_{n,2} = 1$. When $\chi_{n,1} = 1$ and $\chi_{n,2} = 0$, however the left hand side of (3.16) is again dominated by $O_p(1)B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}}$. We note that $\Gamma_n \to I$ uniformly on $[0,1]$ and thus by [A4], $J_n(\Gamma_n) \to J$ (a.s.), pointwise a.e. $|g|$. Since $\xi_{n,n} \to 1$ a.s. and from (3.11) we have for each $t \in (0,1)$, $\chi_{n,2}U_nJ_n(\Gamma_n) \to UJ$ (a.s.), pointwise a.e. $|g|$. Hence the dominated convergence theorem applies and for each $t \in (0,1)$, we obtain

$$\chi_{n,1}(t)\int_t^1 |\chi_{n,2}U_nJ_n(\Gamma_n) - UJ|d|g| \overset{p}{\to} 0 . \tag{3.19}$$

Again for each $t \in (0,1)$, we have using the upper bound in (3.18)

$$IB(\cdot,\underset{\sim}{b})\{\int_t^1 |\chi_{n,2}U_nJ_n(\Gamma_n) - UJ|d|g|\}$$

$$\leq \{I(1 - I)\}^{\delta/2}B(\cdot,\underset{\sim}{b})q_{\frac{1}{4}}\{\int_0^1 B(\cdot,\underset{\sim}{b})q_{\frac{1}{4}}d|g|\}\ O_p(1), \tag{3.20}$$

where the right hand side is a $|g|$-integrable function. It then follows from (3.19) and the dominated convergence theorem that $S_{n,11}^{(1)} \overset{p}{\to} 0$ as $n \to \infty$.

To handle $S_{n,12}^{(1)}$ we write

$$S_{n,12}^{(1)} \leq \int_0^{\xi_{n,1}} IB(\cdot,\underset{\sim}{b})d|g|\{\int_t^1 |U|B(\cdot,\underset{\sim}{b})d|g|\}$$

$$+ \int_{\xi_{n,n}}^1 IB(\cdot,\underset{\sim}{b})d|g|\{\int_t^1 |U|B(\cdot,\underset{\sim}{b})d|g|\} . \tag{3.21}$$

The first term on the right hand side may be bounded by

$$\int_0^{\xi_{n,1}} IB(\cdot,\underset{\sim}{b})d|g|\{\int_t^1 \{I(1 - I)\}^{\frac{1}{4}}B(\cdot,\underset{\sim}{b})q_{\frac{1}{4}}d|g|\}\rho_{q_{\frac{1}{2}}}(U,0)$$

$$\leq (\int_0^{\xi_{n,1}} \{I(1 - I)\}^{\delta/2}B(\cdot,\underset{\sim}{b})q_{\frac{1}{4}}d|g|)\cdot(\int_0^1 B(\cdot,\underset{\sim}{b})q_{\frac{1}{4}}d|g|)O_p(1)$$

$= o_p(1)$, since $\xi_{n,1} \overset{p}{\to} 0$ and the integral converges.
The same argument will also show that the second term on the right hand side of (3.21) is $o_p(1)$. Hence, finally it follows from (3.15) that

$$S_{n,1}^{(1)} \overset{p}{\to} \zeta_1 = \int_0^1 [\int_t^1 UJdg]dL_2, \quad \text{as} \quad n \to \infty. \tag{3.22}$$

Next, we note that by [A3] and the definition of $S_{n,1}^{(2)}$,

$$|S_{n,1}^{(2)}| \leq \int_0^1 IB(\cdot,\underset{\sim}{b})d|g|\{\int_t^1 (1 - I)n^{\frac{1}{2}}|J_n(\Gamma_n) - J(\Gamma_n)|d|g|\}$$

$$\leq (\int_0^1 \{I(1 - I)\}^{\delta/2}B(\cdot,\underset{\sim}{b})q_{\frac{1}{4}}d|g|)(\int_0^1 n^{\frac{1}{2}}|J_n(\Gamma_n) - J(\Gamma_n)|d|g|)$$

$$= O(1)o_p(1). \tag{3.23}$$

To handle $S_{n,1}^{(3)}$ we note that it may be written in the form

$$S_{n,1}^{(3)} = \int_0^1 dL_2(t)\chi_{n,1}(t)\{\int_t^1 \chi_{n,2}(1-I)U_n((J(\Gamma_n) - J)/(\Gamma_n - I))dg\} \tag{3.24}$$

where the indicators $\chi_{n,1}$, $\chi_{n,2}$ were defined preceding (3.15). Define $\zeta_2 = \int_0^1\{\int_t^1 (1 - I)UJ'dg\}dL_2$. Then

$$|S_{n,1}^{(3)} - \zeta_2| \leq S_{n,13}^{(1)} + S_{n,14}^{(1)}, \tag{3.25}$$

where

$$S_{n,13}^{(1)} = \int_0^1 IB(\cdot,\underset{\sim}{b})\chi_{n,1}d|g|\{\int_t^1 (1-I)|\chi_{n,2}U_n(J(\Gamma_n)-J)/(\Gamma_n-I)-UJ'|d|g|\}, \tag{3.26}$$

$$S_{n,14}^{(1)} = \int_0^1 IB(\cdot,\underset{\sim}{b})\overline{\chi}_{n,1}d|g|\{\int_t^1 (1-I)|UJ'|d|g|\}. \tag{3.27}$$

The analysis of $S_{n,13}^{(1)}$ is very similar to that of $S_{n,11}^{(1)}$. Note that $|J(\Gamma_n) - J|/|\Gamma_n - I| \leq B(\Gamma_n,\underset{\sim}{c}) \vee B(I,\underset{\sim}{c})$ by [A4]. Once again since $0 < \Gamma_n < 1$ in the range under consideration in (3.26) we may invoke Theorem 2 of Wellner (1972): for some $A^* \subset \Omega$ with $P(A^*) = 1$, there exists for each $\omega^* \in A^*$, an integer n_{ω^*} such that for $n \geq n_{\omega^*}$

$$|J(\Gamma_n) - J|/|\Gamma_n - I| \leq M_0B(\cdot,\underset{\sim}{c})q_{\frac{1}{2}}/\tilde{q}, \tag{3.28}$$

where M_0 $(< \infty)$ is a constant and $\tilde{q}$ is defined as in (3.17). By steps similar to (3.16) through (3.19) for $S_{n,11}^{(1)}$ and the continuity of J', we obtain, for each $t \in (0,1)$

$$\chi_{n,1}(t)\int_t^1 (1 - I)|\chi_{n,2}U_n(J(\Gamma_n) - J)/(\Gamma_n - I) - UJ'|d|g| \overset{p}{\to} 0 \ . \quad (3.29)$$

Furthermore in view of [A4],

$$IB(\cdot,\underset{\sim}{b})\{\int_t^1 (1 - I)|\chi_{n,2}U_n(J(\Gamma_n) - J)/(\Gamma_n - I) - UJ'|d|g|\}$$

$$\leq \{I(1 - I)\}^{\delta/2}B(\cdot,\underset{\sim}{b})q_{\frac{1}{4}}\{\int_0^1 I(1 - I)B(\cdot,\underset{\sim}{c})q_{\frac{1}{4}}d|g|\}o_p(1) \qquad (3.30)$$

and the right hand side is a $|g|$-integrable function. Hence from (3.29) and the dominated convergence theorem we obtain $S_{n,13}^{(1)} \overset{p}{\to} 0$ as $n \to \infty$. Finally from (3.27) and [A4]

$$S_{n,14}^{(1)} \leq \int_0^{\xi_{n,1}} IB(\cdot,\underset{\sim}{b})d|g|\{\int_t^1 (1 - I)|U|B(\cdot,\underset{\sim}{c})d|g|\}$$

$$+ \int_{\xi_{n,n}}^1 IB(\cdot,\underset{\sim}{b})d|g|\{\int_t^1 (1 - I)|U|B(\cdot,\underset{\sim}{c})d|g|\}, \qquad (3.31)$$

and as in the treatment of (3.21) the first term on the right hand side of (3.31) may be bounded by

$$\int_0^{\xi_{n,1}} IB(\cdot,\underset{\sim}{b})d|g|\{\int_t^1 I^{\frac{1}{4}}(1 - I)^{5/4}B(\cdot,\underset{\sim}{c})q_{\frac{1}{4}}d|g|\}\rho_{q_{\frac{1}{2}}}(U,0)$$

$$\leq (\int_0^{\xi_{n,1}} \{I(1 - I)\}^{\delta/2}B(\cdot,\underset{\sim}{b})q_{\frac{1}{4}}d|g|)(\int_0^1 I(1 - I)B(\cdot,\underset{\sim}{c})q_{\frac{1}{4}}d|g|)O_p(1)$$

$= o_p(1)$, since the integral converges and $\xi_{n,1} \overset{p}{\to} 0$.

The same argument applies to the second term on the right hand side of (3.31) and so we have $S_{n,14}^{(1)} \overset{p}{\to} 0$ and hence finally from (3.25)

$$S_{n,1}^{(3)} \overset{p}{\to} \zeta_2 = \int_0^1 \{\int_t^1 (1 - I)UJ'dg\}dL_2 \ . \qquad (3.32)$$

In the Appendix, Lemma 1, we show $S_{n,1}^{(4)} \overset{p}{\to} 0$ as $n \to \infty$. Thus from (3.14), (3.22), (3.23), (3.32) and the above, it follows that

$$S_{n,1} \xrightarrow{P} \zeta_2 - \zeta_1 \quad \text{as} \quad n \to \infty. \tag{3.33}$$

The analysis of $S_{n,2}$ is entirely analogous, and hence, in the interest of brevity, we omit the details and present only the final result:

$$S_{n,2} \xrightarrow{P} \zeta_3 + \zeta_4 \quad \text{as} \quad n \to \infty, \tag{3.34}$$

where

$$\zeta_3 = \int_0^1 L_1 UJdq \quad \text{and} \quad \zeta_4 = \int_0^1 IL_1 UJ'dq \; . \tag{3.35}$$

Finally, a very similar analysis leads to the conclusion that

$$R_n \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty. \tag{3.36}$$

Hence from (3.7), (3.33), (3.34) and (3.36) we obtain that

$$n^{\frac{1}{2}}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{P} \tfrac{1}{2}(\zeta_2 - \zeta_1 + \zeta_3 + \zeta_4) \quad \text{as} \quad n \to \infty. \tag{3.37}$$

In Lemma 2 of the Appendix we show that

$$\tfrac{1}{2}(\zeta_2 - \zeta_1 + \zeta_3 + \zeta_4) = \int_0^1 UL_0 dq = S, \text{ say}, \tag{3.38}$$

where L_0 is defined by (2.13). Therefore with γ^2 defined by (2.14), S has the normal distribution with mean 0 and variance γ^2. Q.E.D.

The proof of Theorem 2 proceeds very much along the same lines and so we omit some details here. Corresponding to (2.12) and (3.5) we define for each $n > 1$,

$$L^*_{n,1}(t) = 2\int_t^{n^{-1}\tau_n} (I - \Gamma_n)J_n(\Gamma_n)dq, \text{ for } t \in [\zeta_{n,1}, n^{-1}\tau_n] \tag{3.39}$$

with $L^*_{n,1}$ set equal to zero otherwise and

$$L^*_{1,n}(t) = 2\int_t^{n^{-1}\tau_n} J_{(2)}dq, \text{ for } t \in (0, n^{-1}\tau_n], \tag{3.40}$$

with $L^*_{1,n}$ set equal to zero otherwise.

For simplicity we shall write L_1^* for $L_{1,n}^*$ in the sequel. Now

$$\sigma_n^{*2} = \tfrac{1}{2} \int_0^1 L_{n,1}^* dL_{n,2} \quad \text{and} \quad \sigma^2(n^{-1}\tau_n) = 1/2 \int_0^1 L_1^* dL_2. \qquad (3.41)$$

Therefore corresponding to (3.7) through (3.10) we have

$$n^{\frac{1}{2}}(\sigma_n^{*2} - \sigma^2(n^{-1}\tau_n)) = \tfrac{1}{2}(S_{n,1}^* + S_{n,2}^* + R_n^*) \qquad (3.42)$$

where

$$S_{n,1}^* = \int_0^{n^{-1}\tau_n} n^{\frac{1}{2}}(L_{n,1}^* - L_1^*)dL_2 , \qquad (3.43)$$

$$S_{n,2}^* = \int_0^{n^{-1}\tau_n} L_1^* d\{n^{\frac{1}{2}}(L_{n,2} - L_2)\}, \qquad (3.44)$$

$$R_n^* = \int_0^{n^{-1}\tau_n} n^{\frac{1}{2}}(L_{n,1}^* - L_1^*)d(L_{n,2} - L_2). \qquad (3.45)$$

The decomposition corresponding to (3.12) reads

$$\begin{aligned}\tfrac{1}{2}(L_{n,1}^* - L_1^*) &= -\int_t^{n^{-1}\tau_n} U_n J_n(\Gamma_n)dg + \int_t^{n^{-1}\tau_n} (1 - I)n^{\frac{1}{2}}(J_n(\Gamma_n) - J(\Gamma_n))dg \\ &\quad + \int_t^{n^{-1}\tau_n} (1 - I)n^{\frac{1}{2}}(J(\Gamma_n) - J)dg, \quad t \in [\xi_{n,1}, n^{-1}\tau_n) \\ &= -n^{\frac{1}{2}} \int_t^{n^{-1}\tau_n} (1 - I)Jdg, \quad t \in (0, \xi_{n,1}] . \end{aligned} \qquad (3.46)$$

Since $n^{-1}\tau_n \xrightarrow{p} \alpha$ (< 1) by assumption and $\xi_{n,n} \xrightarrow{p} 1$, the set on which $n^{-1}\tau_n < \xi_{n,n}$ has probability which tends to one as $n \to \infty$. The argument used to examine (3.14) now applies with only minor modifications. For instance

$$\int_{\xi_{n,1}}^{n^{-1}\tau_n} dL_2 \{\int_t^{n^{-1}\tau_n} U_n J_n(\Gamma_n)dg - \int_t^{n^{-1}\tau_n} UJdg\} \xrightarrow{p} 0, \qquad (3.47)$$

and the usual argument shows that provided $n^{-1}\tau_n \xrightarrow{p} \alpha$

$$\int_{\xi_{n,1}}^{n^{-1}\tau_n} dL_2\{\int_t^{n^{-1}\tau_n} UJdg\} \xrightarrow{p} \zeta_1^* = \int_0^\alpha dL_2\{\int_t^\alpha UJdg\}. \tag{3.48}$$

Likewise $\int_{\xi_{n,1}}^{n^{-1}\tau_n} dL_2\{\int_t^{n^{-1}\tau_n} (1 - I)n^{\frac{1}{2}}(J_n(\Gamma_n) - J(\Gamma_n)dg\} \xrightarrow{p} 0$ and

$$\int_{\xi_{n,1}}^{n^{-1}\tau_n} dL_2\{\int_t^{n^{-1}\tau_n} (1 - I)n^{\frac{1}{2}}(J(\Gamma_n) - J)dg\} \xrightarrow{p} \zeta_2^* \tag{3.49}$$

where

$$\zeta_2^* = \int_0^\alpha dL_2\{\int_t^\alpha (1 - I)UJ'dg\} . \tag{3.50}$$

Finally $n^{\frac{1}{2}}\int_0^{\xi_{n,1}} dL_2\{\int_t^{n^{-1}\tau_n} J_{(2)}dg\} \xrightarrow{p} 0$, by the argument in Lemma 1 of the Appendix. From (3.46) through (3.49) we obtain

$$S_{n,1}^* \xrightarrow{p} \zeta_2^* - \zeta_1^* . \tag{3.51}$$

Finally for $S_{n,2}^*$ and R_n^* the results are

$$S_{n,2}^* \xrightarrow{p} \zeta_3^* + \zeta_4^* \quad \text{and} \quad R_n^* \xrightarrow{p} 0 \tag{3.52}$$

where

$$\zeta_3^* = \int_0^\alpha L_1^+UJdg \quad \text{and} \quad \zeta_4^* = \int_0^\alpha IL_1^+UJdg . \tag{3.53}$$

Hence from (3.42), (3.51) and (3.52) we get

$$n^{\frac{1}{2}}(\sigma_n^{*2} - \sigma^2(n^{-1}\tau_n)) \xrightarrow{p} \tfrac{1}{2}(\zeta_2^* - \zeta_1^* + \zeta_3^* + \zeta_4^*).$$

A minor modification of Lemma 2 of the Appendix will show that

$$\tfrac{1}{2}(\zeta_2^* - \zeta_1^* + \zeta_3^* + \zeta_4^*) = \int_0^\alpha UL_0^+dg,$$

and thus the first part of Theorem 2 is proven. For the second part we need only recognize that for $n^{-1}\tau_n < \alpha$,

$$n^{1/2}(\sigma^2(n^{-1}\tau_n) - \sigma^2(\alpha)) = -2n^{1/2}\left(\int_0^{n^{-1}\tau_n} J_{(1)}dg\right)\left(\int_{n^{-1}\tau_n}^{\alpha} J_{(2)}dg\right) - 1/2\, n^{1/2}\int_{n^{-1}\tau_n}^{\alpha} L_1^+ dL_2$$

with a similar expression if $n^{-1}\tau_n > \alpha$. Now, for the first term we may use the argument of Lemma 1 to show that it converges to zero in probability as $n \to \infty$. For the second term the additional assumptions on g and τ_n gives

$$\begin{aligned} \left|n^{1/2}\int_{n^{-1}\tau_n}^{\alpha} L_1^+ dL_2\right| &\le O(1)n^{1/2}|g(n^{-1}\tau_n) - g(\alpha)| \\ &\le O(1)n^{1/2}|n^{-1}\tau_n - \alpha|\,|g'(\alpha) + o_p(1)| \\ &= o_p(1). \end{aligned}$$

Hence Theorem 2 is proven.

4. Some general remarks.

Consider the following example. Let $X_1,\ldots,X_n$ be iid rv's with df F and $E|X_1|^r < \infty$ for some $r > 4$. We shall consider the sample variance

$$\hat{\sigma}_n^2 = n^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \tag{4.1}$$

where $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ is the sample mean. Then in the notation of Theorem 1 we have $c_{n,i} = 1$ for all i, $1 \le i \le n$ and $g = F^{-1}$. Note that $E|X_1|^r < \infty$ implies $\lim_{t\to 0+,1-} t(1-t)|F^{-1}(t)|^r = 0$. Thus $|g| \le D$ on $(0,1)$ with $a_1 = a_2 = 1/r$. Also $J = J_n = 1$ so that $b_1 = b_2 = 0$ and A3, A4 hold trivially. Therefore if we take δ such that $1/r = 1/4 - \delta$ we have $\delta > 0$ provided $r > 4$. By an integration by parts (2.15) holds.

Now L_0 of (2.13) reduces to

$$L_0(t) = -2F^{-1}(t) + 2\int_0^1 F^{-1}(s)ds.$$

For simplicity let us take $EX_1 = 0$ so that $\int_0^1 F^{-1}(s)ds = 0$. Then in (2.17) we may take $G_0(t) = -\{F^{-1}(t)\}^2$. So we obtain $\gamma^2 = \mu_4 - \sigma^4$ where $\mu_4 = EX_1^4$ and $\sigma^2 = EX_1^2 = \sigma^2(1)$. Hence Theorem 1 yields

$$n^{\frac{1}{2}}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow[\mathcal{L}]{} N(0, \mu_4 - \sigma^4), \tag{4.2}$$

a result which is obtainable under the assumption $r = 4$. In this context Theorem 1 "just fails" to yield the slightly stronger conclusion.

The above example presents a very interesting observation pertaining to the different sets of conditions that suffice to yield the almost sure (a.s.) convergence of the statistics $T_{n,n}$, their asymptotic normality and the asymptotic normality of the estimator $\hat{\sigma}_n^2$ of the asymptotic variance. We have noted that if A1, A2 obtain and (2.7) holds then

$$\int_0^1 B(\cdot, \underset{\sim}{b})q_{\frac{1}{2}}d|g| < \infty \tag{4.3}$$

and both $\mu_n(1)$ of (2.5) and $\sigma^2(1)$ of (2.6) are finite and

$$n^{\frac{1}{2}}(T_{n,n} - \mu_n(1)) \xrightarrow[\mathcal{L}]{} N(0, \sigma^2(1)). \tag{4.4}$$

Under A1, A2 (2.7) ensures the finiteness of $\mu_n(1)$ and (4.3) that of $\sigma^2(1)$. To obtain the asymptotic normality of the variance estimator of $\hat{\sigma}_n^2(1)$ of (2.8) we impose the additional assumptions A3, A4 and replace (4.3) by the stronger condition (2.15). The a.s. convergence of $\dot{T}_{n,n}$ has been studied by Wellner (1977). If A1, A2 obtain and $a_1 + b_1 = a_2 + b_2 = 1-\delta$ then $\mu_n(1)$ is finite and

$$(T_{n,n} - \mu_n) \underset{a.s.}{\to} 0. \tag{4.5}$$

Sen (1978) has obtained the a.s. convergence of $\hat{\sigma}_n^2(1)$ following a different technique.

For the "stopped statistics" T_{n,τ_n} their asymptotic normality is derived in [2]. Again if A1, A2 obtain and (2.7) holds

$$n^{\frac{1}{2}}(T_{n,\tau_n} - \mu_n(n^{-1}\tau_n)) \underset{L}{\to} N(0, \sigma^2(\alpha)) \tag{4.6}$$

provided $n^{-1}\tau_n \overset{P}{\to} \alpha \in (0,1)$ and g admits a derivative at α or $n^{\frac{1}{2}}(n^{-1}\tau_n - \alpha) = 0_p(1)$ and g is continuous at α. In the latter case if we further assume the stronger condition $n^{\frac{1}{2}}(n^{-1}\tau_n - \alpha) \overset{P}{\to} 0$ then $\mu_n(n^{-1}\tau_n)$ in (4.6) can be also replaced by $\mu_n(\alpha)$.

The a.s. convergence of T_{n,τ_n} can be discussed along the lines of Wellner (1977) assuming $n^{-1}\tau_n \underset{a.s.}{\to} \alpha$.

5. Appendix

Lemma 1: Under the hypothesis of Theorem 1 and $S_{n,1}^{(4)}$ defined by (3.12) and (3.14), we have $S_{n,1}^{(4)} \overset{P}{\to} 0$ as $n \to \infty$.

Proof. We first write $S_{n,1}^{(4)}$ in the form

$$S_{n,1}^{(4)} = -(S_{n,11}^{(4)} + S_{n,12}^{(4)} + S_{n,13}^{(4)}) \tag{5.1}$$

where

$$S_{n,11}^{(4)} = (n^{\frac{1}{2}}\int_{\xi_{n,1}}^{\xi_{n,n}} IJdq)(\int_{\xi_{n,n}}^{1} (1 - I)Jdq),$$

$$S_{n,12}^{(4)} = n^{\frac{1}{2}}\int_{0}^{\xi_{n,1}} IJdq\{\int_{t}^{1} (1 - I)Jdq\},$$

$$S_{n,13}^{(4)} = n^{\frac{1}{2}}\int_{\xi_{n,n}}^{1} IJdq\{\int_{t}^{1} (1 - I)Jdq\}.$$

Therefore,

$$|S_{n,11}^{(4)}| \leq (n^{\frac{1}{2}}\int_{\xi_{n,1}}^{\xi_{n,n}} IB(\cdot,\underset{\sim}{b})d|g|)(\int_{\xi_{n,n}}^{1} (1 - I)B(\cdot,\underset{\sim}{b})d|g|). \tag{5.2}$$

Now

$$\int_{\xi_{n,1}}^{\xi_{n,n}} IB(\cdot,\underset{\sim}{b})d|g| \leq \xi_{n,n}^{3/4} \int_{\xi_{n,1}}^{\xi_{n,n}} I^{\frac{1}{2}}B(\cdot,\underset{\sim}{b})d|g|, \tag{5.3}$$

and

$$\int_{\xi_{n,n}}^{1} (1 - I)B(\cdot,\underset{\sim}{b})d|g| \leq (1 - \xi_{n,n})^{\frac{1}{2}}\int_{\xi_{n,n}}^{1} I^{\delta/2-\frac{1}{2}}(1 - I)^{\delta/2+\frac{1}{2}}B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}}d|g|. \tag{5.4}$$

and therefore from (5.2),

$$|S_{n,11}^{(4)}| \leq n^{\frac{1}{2}} \int_{\xi_{n,1}}^{\xi_{n,n}}\{I(1 - I)\}^{\frac{1}{2}}B(\cdot,\underset{\sim}{b})d|g|\int_{\xi_{n,n}}^{1} \{I(1 - I)\}^{\delta/2+\frac{1}{2}}B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}}d|g|$$

$$\leq (\int_{\xi_{n,1}}^{\xi_{n,n}}\{I(1 - I)\}^{\delta/2}B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}}d|g|)(\int_{\xi_{n,n}}^{1} B(\cdot,\underset{\sim}{b})q_{\frac{1}{2}}d|g|)n^{\frac{1}{2}}(1 - \xi_{n,n})^{\frac{1}{2}}$$

$$= O_p(1)o_p(1)O_p(1) = o_p(1).$$

The argument for $S_{n,13}^{(4)}$ is entirely analogous while for $S_{n,12}^{(4)}$ the steps are similar except that we use $n\xi_{n,1} = O_p(1)$. Hence $S_{n,1i}^{(4)} \xrightarrow{p} 0$ for $i = 1,2,3$ and the lemma follows from (5.1).

Lemma 2. With $\zeta_1,\zeta_2,\zeta_3,\zeta_4$ defined as in (3.22), (3.32) and (3.35) equation (3.38) holds.

Proof. From (2.11) and (3.35) we have

$$\zeta_3 + \zeta_4 = \int_0^1 L_1UJ'_{(1)}dg. \tag{5.5}$$

Also from (3.22) and (3.32)

$$\zeta_2 - \zeta_1 = \int_0^1 dL_2\{\int_t^1 UJ'_{(2)}dg\}. \tag{5.6}$$

Integrating by parts, we obtain

$$\zeta_2 - \zeta_1 = \int_0^1 L_2 UJ'_{(2)} dg + L_2(t)\{\int_t^1 UJ'_{(2)} dg\}\Big|_{t=0}^{t=1}.$$

We shall show

$$\lim L_2(t) \int_t^1 UJ'_{(2)} dg = 0 \tag{5.7}$$

where the limit is taken in each of the two cases $t \to 0+$ and $t \to 1-$. In what follows this is to be interpreted whenever the limit is not explicitly stated.

Now for each $t \in (0,1)$

$$|L_2(t)\int_t^1 UJdg| \le 2\rho_{q_{l_2}}(U,0)(\int_0^t IB(\cdot,\underset{\sim}{b})d|g|)(\int_t^1 B(\cdot,\underset{\sim}{b})q_{l_2}d|g|). \tag{5.8}$$

In view of relations similar to (5.3) and (5.4) the function on the right hand side of (5.8) is dominated by

$$(\int_0^t \{I(1-I)\}^{\delta/2} B(\cdot,\underset{\sim}{b})q_{l_2}d|g|)(\int_t^1 B(\cdot,\underset{\sim}{b})q_{l_2}d|g|), \tag{5.9}$$

and so by (2.15), (5.9) must vanish in the limit as $t \to 0+$ and $t \to 1-$. Hence

$$\lim L_2(t)\int_t^1 UJdg = 0 . \tag{5.10}$$

Again for each $t \in (0,1)$

$$|L_2(t)\int_t^1 (1-I)UJ'dg| \le 2\rho_{q_{l_2}}(U,0)(\int_0^t IB(\cdot,\underset{\sim}{b})d|g|)(\int_t^1 (1-I)B(\cdot,\underset{\sim}{c})q_{l_2}d|g|). \tag{5.11}$$

and the function on the right hand side of (5.11) is dominated by

$$(\int_0^t \{I(1-I)\}^{\delta/2} B(\cdot,\underset{\sim}{b})q_{l_2}d|g|)(\int_t^1 I(1-I)B(\cdot,\underset{\sim}{c})q_{l_2}d|g|). \tag{5.12}$$

It follows from (2.10) and (2.15) that (5.12) vanishes in the limit as $t \to 0+$ and $t \to 1-$. So

$$\lim L_2(t)\int_t^1 (1 - I)UJ'dq = 0. \tag{5.13}$$

But (5.10) and (5.13) imply (5.7) and therefore $\zeta_2 - \zeta_1 = \int_0^1 L_2UJ'_{(2)}dq$. Hence using (5.5) and (2.13),

$$\tfrac{1}{2}(\zeta_2 - \zeta_1 + \zeta_3 + \zeta_4) = \int_0^1 UL_0dq,$$

which is (3.38).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
An estimator of the asymptotic variance of (a randomly stopped) linear combination of a function of order statistics is considered and its asymptotic normality is studied under appropriate regularity conditions. A comparative study of the regularity conditions pertaining to the asymptotic normality and strong convergence of linear combinations of functions or order statistics and their estimated asymptotic variances is also made.

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